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Eta invariants with spectral boundary conditions

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Abstract
We study the asymptotics of the heat trace $\mathrm{Tr}\{fP\,\mathrm{e}^{-tP^2}\}$ where P is an operator of Dirac type, where f is an auxiliary smooth smearing function which is used to localize the problem, and where we impose spectral boundary conditions. Using functorial techniques and special case calculations, the boundary part of the leading coefficients in the asymptotic expansion is found.

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1. Introduction

Let P be an operator of Dirac type with leading symbol γ on a vector bundle V over a compact m-dimensional Riemannian manifold M with smooth boundary ∂M. One may choose a Hermitian inner product $(\cdot,\cdot)$ and a Hermitian connection ∇ on V so that γ is skew-adjoint and so that $\nabla\gamma = 0$ [11]; such structures are said to be *compatible* with the given Clifford module structure γ. Let indices i, j range from 1 to m and index a local orthonormal frame $\{e_i\}$ for the tangent bundle of M. We adopt the Einstein convention and sum over repeated indices to expand

$$P = \gamma_i \nabla_{e_i} + \psi_P$$

where ψ_P is a smooth endomorphism of V; the sign convention for ψ_P differs from that in [11, 12]. Note that the matrices γ_i are skew-adjoint endomorphisms of V satisfying the *Clifford commutation relations*

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}.$$

If ∂M is non-empty, then we must impose suitable boundary conditions. For m even, P always admits local elliptic boundary conditions; see, for example, the discussion of *bag boundary conditions* in [7, 8]. However, if m is odd, there is a topological obstruction to the existence of local boundary conditions for certain operators. We therefore introduce *spectral* *boundary conditions*; these boundary conditions, which are defined regardless of the parity of m, play a crucial role in the index theorem for manifolds with boundary [3].

Spectral boundary conditions were first introduced by Atiyah *et al* [3] in their study of the Hirzebruch signature theorem for manifolds with non-empty boundary. The crucial point at issue was the definition of a suitable elliptic boundary value problem for the signature operator whose index was the signature of the manifold. Although the de Rham complex, whose index is the Euler characteristic, admits local boundary conditions (i.e. boundary conditions which are a mixture of Robin and Dirichlet), the signature complex does not. The signature complex does admit spectral boundary conditions—these are pseudo-differential boundary conditions—and their introduction was a crucial turning point.

In order to describe these boundary conditions, near the boundary we choose a local orthonormal frame so e_m is the inward unit geodesic normal vector field and $\{e_a\}$ for $1 \leqslant a \leqslant m-1$ is the induced orthonormal frame for the tangent bundle of the boundary. Let

$$\gamma_a^T := -\gamma_m \gamma_a$$

be the induced tangential Clifford module structure. Let ψ_A be an auxiliary smooth endomorphism of $V|_{\partial M}$. Consider the auxiliary operator of Dirac type on $V|_{\partial M}$

$$A := \gamma_a^T \nabla_{e_a} + \psi_A.$$

Assume A has no purely imaginary eigensections. Let C be a suitable contour in the complex plane containing the spectrum of A with positive real part. Let

$$\Pi_A^+ := \frac{1}{2\pi\sqrt{-1}} \int_C (A-\lambda)^{-1}\,\mathrm{d}\lambda$$

be the *spectral boundary operator*; Π_A^+ is spectral projection on the generalized eigenspaces associated to eigenvalues with positive real part. Let P_A be the realization of P with respect to the boundary conditions defined by Π_A^+.

The spectral information regarding this boundary value problem is encoded in the zeta function and the eta function which are defined as follows. Assume for the sake of simplicity that P_A is self-adjoint (we will be forced to drop this requirement presently). Let (λ_l, φ_l) be a spectral resolution of P_A; $\{\varphi_l\}$ is a complete orthonormal basis for $L^2(V)$ such that

$$P_A \varphi_l = \lambda_l \varphi_l, \qquad \Pi_A^+ \varphi_l\big|_{\partial M} = 0.$$

Then the zeta function associated with P_A^2 is

$$\zeta(s; P, A) := \sum_{\lambda_l \neq 0} \left(\lambda_l^2\right)^{-s} \tag{1a}$$

valid for $\Re s > m/2$. Note, the fact that P_A as a first order differentiable operator can have positive and negative eigenvalues does not enter the zeta function of the Laplace-type operator P_A^2. However, the sign is taken into account defining the eta function of P_A

$$\eta(s; P, A) := \sum_{\lambda_l \neq 0} \mathrm{sign}(\lambda_l)|\lambda_l|^{-s}, \tag{1b}$$

valid for $\Re s > m-1$. Similarly, one can define $\zeta(s; A)$ and $\eta(s; A)$; since ∂M is closed, there is no boundary condition required.

Although the above series representations (1*a*) and (1*b*) are valid only in the given region of the complex s-plane, the eta and zeta functions can be analytically continued to meromorphic functions defined on the whole complex plane. The value $\eta(0; P, A)$ is essential for the description of the index of P_A.

One can also discuss the heat trace. Let ϕ be the 'initial temperature distribution' and let $u_\phi(t,x)$ denote the subsequent temperature distribution. Then $u_\phi(t,x)$ is determined by the equations

$$(\partial_t + P^2)u_\phi(t,x) = 0, \qquad \Pi_A^+ u_\phi\big|_{\partial M} = 0 \qquad \text{and} \qquad u_\phi(0,x) = \phi(x).$$

The associated fundamental solution $\mathcal{K} : \phi \to u_\phi$ is then given by $\mathcal{K} = \mathrm{e}^{-tP_A^2}$. Let $\mathrm{d}x$ and $\mathrm{d}y$ be the Riemannian measures on M and on ∂M respectively. There exists a smooth endomorphism-valued kernel $K(t,x,\bar{x},P^2,A) : V_{\bar{x}} \to V_x$ such that

$$u_\phi(t,x) = (\mathcal{K}\phi)(t,x) = \int_M K(t,x,\bar{x},P^2,A)\phi(\bar{x})\,\mathrm{d}\bar{x}.$$

For fixed t, the operator $\mathcal{K}(t) : \phi \to \phi(t,\cdot)$ is of trace class. For $F \in C^\infty(\mathrm{End}(V))$ a smooth auxiliary smearing endomorphism used for localizing the problem, we define

$$a^\zeta(F,P,A) := \mathrm{Tr}_{L^2}\big(F\,\mathrm{e}^{-tP_A^2}\big) = \int_M \mathrm{Tr}_{V_x}(F(x)K(t,x,x,P^2,A))\,\mathrm{d}x$$

$$a^\eta(F,P,A) := \mathrm{Tr}_{L^2}\big(FP_A\,\mathrm{e}^{-tP_A^2}\big) = \int_M \mathrm{Tr}_{V_x}(F(x)P_AK(t,x,x,P^2,A))\,\mathrm{d}x.$$

Grubb and Seeley [25] showed that there are asymptotic expansions as $t \downarrow 0^+$ of the form

$$\begin{aligned} a^\zeta(F,P,A) &\sim \sum_{n=0}^{m-1} a_n^\zeta(F,P,A)t^{(n-m)/2} + \mathcal{O}(\ln t),\\ a^\eta(F,P,A) &\sim \sum_{n=0}^{m-1} a_n^\eta(F,P,A)t^{(n-m-1)/2} + \mathcal{O}(t^{1/2}\ln t). \end{aligned} \tag{1c}$$

We refer to the coefficients a_n^ζ and a_n^η as the zeta and eta invariants respectively.

We note that there are in fact full asymptotic series for a^ζ and a^η. However non-local terms and log terms arise when $n \geqslant m$. Since we shall assume that $n < m$, these terms play no role for us. We shall normally assume that $F = f \cdot \mathrm{Id}$ where $f \in C^\infty(M)$ is scalar valued, but it will be convenient occasionally to have this more general setting available.

The Mellin transform can be used to relate the zeta and eta functions and the small-t asymptotic expansion of the heat-trace. For $f = 1$, one has [21, 41]

$$\mathrm{Res}\ \zeta\left(\frac{m-n}{2}; P, A\right) = \frac{a_n^\zeta(1,P,A)}{\Gamma\left(\frac{m-n}{2}\right)},$$

$$\mathrm{Res}\ \eta(m-n;P,A) = \frac{2a_n^\eta(1,P,A)}{\Gamma\left(\frac{m-n+1}{2}\right)}.$$

Similar formulae hold for the general endomorphism F; this will play an important role in our subsequent development.

The heat trace coefficients a_n^ζ and a_n^η of equation (1c) are locally computable for $n < m$; they play a crucial role in many areas. For example, the particular coefficient a_m^ζ is relevant in the quantum mechanics of closed cosmologies, where it describes how quantum effects modify the behaviour of the universe near classical singularities [14, 18, 40]. More generally, the leading coefficients a_n^ζ, $n = 0, 1, \ldots, m$ are needed in different quantum field theories. These theories are generically plagued by divergences which are removed by a renormalization. In the zeta function scheme [17], as well as in the framework of recent developments of algebraic quantum field theory [33], at one-loop, divergences are completely described by the leading coefficients. As a result, their knowledge is equivalent to a knowledge of the one-loop renormalization group equations [43], which provides one reason for the consideration of heat kernel coefficients in physics. In addition, if an exact evaluation of relevant quantities is not possible, asymptotic expansions are often very useful and most suitably given in terms of heat kernel coefficients [5, 15]. In this context of quantum field theories, apart from cosmology, spectral boundary conditions most prominently make their appearance in bag models where they have important advantages over local elliptic boundary conditions. In particular, it is the only self-adjoint boundary condition which respects the charge conjugation property and the so-called γ_5 symmetry [19, 27, 34]. In Euclidean gauge field theories, this property enables one to consider a compactified Dirac problem where spectral information such as functional determinants are directly related to the original problem [28, 42].

Whereas the above relates to a_n^ζ, the η-function arises in the analysis of fermion number fractionization in different field theory models [35, 36, 39]. The fermion number N is a transcendental function of the parameters of the theory and is related to $\eta(0; H, A)$ of the pertinent Dirac Hamiltonian H and boundary operator A. In a simplified consideration [30] the fermion number of the vacuum will be formally obtained by filling the Dirac sea,

$$\begin{aligned} N &= [\text{number of negative-energy states of } H] \\ &= \tfrac{1}{2}\{[(\text{numb. of pos.-en. states of } H) + (\text{numb. of neg.-en. states of } H)] \\ &\quad - [(\text{numb. of pos.-en. states of } H) - (\text{numb. of neg.-en. states of } H)]\} \end{aligned}$$

Regularizing this divergent expression it becomes $(1/2)[0 - \eta(0; H, A)]$. A rigorous proof can be found in [31].

Furthermore, interpreting (1*a*) and (1*b*) as a moment problem for the spectral density function, even and odd part of the density can be found provided $\zeta(s; H, A)$ and $\eta(s; H, A)$ can be evaluated [37]. Knowledge of the leading coefficients a_n^ζ respectively a_n^η amounts to an asymptotic knowledge of the even and odd part of the density for large eigenvalues $|\lambda_l|$ opening up the possibility for the approximate evaluation of different quantities in quantum field theories as for example the finite temperature induced fermion number [36].

The invariants a_n^ζ have been studied extensively [16, 23, 25, 26]; the invariants a_n^η have received a bit less attention. We may decompose

$$a_n^\zeta(F, P, A) = a_n^{\zeta,M}(F, P) + a_n^{\zeta,\partial M}(F, P, A),$$

and

$$a_n^\eta(F, P, A) = a_n^{\eta,M}(F, P) + a_n^{\eta,\partial M}(F, P, A)$$

as the sum of interior and boundary contributions. There exist local endomorphism valued invariants $\mathrm{e}_n^{\zeta,M}(x, P)$ and $\mathrm{e}_n^{\eta,M}(x, P)$, which are homogeneous of weight n in the jets of the total symbol of P, so that

$$a_n^{\zeta,M}(F, P) = \int_M \mathrm{Tr}\big\{F(x)\,\mathrm{e}_n^{\zeta,M}(x, P)\big\}\,\mathrm{d}x,$$

and

$$a_n^{\eta,M}(F, P) = \int_M \mathrm{Tr}\big\{F(x)\,\mathrm{e}_n^{\eta,M}(x, P)\big\}\,\mathrm{d}x.$$

We note that there is a parity constraint for the interior invariants

$$a_n^{\zeta,M} = 0 \quad \text{if} \quad n \quad \text{is odd} \qquad \text{and} \qquad a_n^{\eta,M} = 0 \quad \text{if} \quad n \text{ is even.}$$

Formulae for the invariants $a_n^{\zeta,M}$ for $n = 0, 2, 4, 6, 8$ follow from work of [2, 4, 20, 32]; similar formulae for the invariants $a_n^{\eta,M}$ are known for $n = 1, 3$ [11].

Let $\nabla_m^k F$ denote the kth normal covariant derivative of the endomorphism F. There are local invariants $\mathrm{e}_{n,k}^{\zeta,\partial M}(y, P, A)$ and $\mathrm{e}_{n,k}^{\eta,\partial M}(y, P, A)$ which are homogeneous of weight $n-k-1$

in the jets of the total symbol of P and of A so that

$$a_n^{\zeta,\partial M}(F,P,A)=\sum_{k<n}\int_{\partial M}\mathrm{Tr}\{\nabla_m^k F(y)\cdot \mathrm{e}_{n,k}^{\zeta,\partial M}(y,P,A)\}\,\mathrm{d}y$$

and

$$a_n^{\eta,\partial M}(F,P,A)=\sum_{k<n}\int_{\partial M}\mathrm{Tr}\{\nabla_m^k F(y)\cdot \mathrm{e}_{n,k}^{\eta,\partial M}(y,P,A)\}\,\mathrm{d}y.$$

Let Ω_{ij} be the curvature of the connection ∇. We define

$$W_{ij}:=\Omega_{ij}-\frac{1}{4}R_{ijkl}\gamma_k\gamma_\ell,\qquad \beta(m):=\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{1}{2}\right)^{-1}\Gamma\left(\frac{m+1}{2}\right)^{-1},$$

and

$$E:=\frac{1}{2}(\psi_{P;i}\gamma_i-\gamma_i\psi_{P;i})-\psi_P^2-\frac{1}{4}(\psi_P\gamma_i+\gamma_i\psi_P)(\psi_P\gamma_i+\gamma_i\psi_P)-\frac{1}{2}\gamma_i\gamma_j W_{ij}-\tfrac{1}{4}\tau.$$

Let $\tau:=R_{ijji}$ be the scalar curvature and let L_{ab} be the second fundamental form. We can use [11, 16, 23] to see

Theorem 1.1. *If P_A is self-adjoint, if A is self-adjoint, and if $F=f\cdot\mathrm{Id}$ is scalar,*

(i) $a_0^\zeta(F,P,A)=(4\pi)^{-m/2}\int_M f\,\mathrm{Tr}\{\mathrm{Id}\}\,\mathrm{d}x$.

(ii) If $m\geqslant 2$, $a_1^\zeta(F,P,A)=(4\pi)^{-(m-1)/2}\frac{1}{4}(\beta(m)-1)\int_{\partial M} f\,\mathrm{Tr}\{\mathrm{Id}\}\,\mathrm{d}y$.

(iii) If $m\geqslant 3$, $a_2^\zeta(F,P,A)=(4\pi)^{-m/2}\int_M f\,\mathrm{Tr}\{\frac{1}{6}\tau\mathrm{Id}+E\}\,\mathrm{d}x$
$+(4\pi)^{-m/2}\int_{\partial M}\{\frac{1}{3}(1-\frac{3}{4}\pi\beta(m))L_{aa}f-\frac{m-1}{2(m-2)}(1-\frac{1}{2}\pi\beta(m))f_{;m}\}\,\mathrm{Tr}\{\mathrm{Id}\}\,\mathrm{d}y$.

We refer to [23] for the corresponding computation of $a_3^\zeta(f,D,\mathcal{B})$. In this paper, we establish formulae for a_n^η without self-adjointness assumptions:

Theorem 1.2. *Let* $F=f\cdot\mathrm{Id}$ *be scalar, then*

(i) $a_0^\eta(F,P,A)=0$.

(ii) If $m\geqslant 2$, $a_1^\eta(F,P,A)=(4\pi)^{-m/2}(1-m)\int_M f\,\mathrm{Tr}\{\psi_P\}\,\mathrm{d}x$.

(iii) If $m\geqslant 3$, $a_2^\eta(F,P,A)=(4\pi)^{-(m-1)/2}\int_{\partial M} f\,\mathrm{Tr}\{\frac{2-m}{4}(\beta(m)-1)\psi_P-\frac{1}{4}\beta(m)\gamma_m\psi_A\}\,\mathrm{d}y$.

(iv) If $m\geqslant 4$, $a_3^\eta(F,P,A)=-\frac{1}{12}(4\pi)^{-m/2}\int_M f\,\mathrm{Tr}\{[2(m-1)\psi_{P;i}+3(4-m)\psi_P\gamma_i\psi_P$
$+3\gamma_j\psi_P\gamma_j\gamma_i\psi_P]_{;i}+(3-m)\{\tau\psi_P+6\gamma_i\gamma_j W_{ij}\psi_P-6\psi_P\psi_{P;i}\gamma_i+(4-m)\psi_P\psi_P\psi_P$
$+3\psi_P\psi_P\gamma_i\psi_P\gamma_i\}\}\,\mathrm{d}x+(4\pi)^{-m/2}\int_{\partial M}\mathrm{Tr}\{\frac{(m-3)(m-1)}{2(m-2)}(1-\frac{1}{2}\pi\beta(m))f_{;m}\psi_P$
$-f\frac{(3-m)^2}{4(m-2)}(\psi_P\psi_A+\gamma_m\psi_P\gamma_m\psi_A)+f\frac{3-m}{3}(1-\frac{3}{4}\pi\beta(m))L_{aa}\psi_P$
$+f\{\frac{(m-3)(m-1)}{2(m-2)}(1-\frac{1}{2}\pi\beta(m))-\frac{1}{6}(m-1)\}\psi_{P;m}$
$-f\frac{3-m}{4(m-2)}(\gamma_a^T\psi_P\gamma_a^T\psi_A-\gamma_a\psi_P\gamma_a\psi_A+2\gamma_m\gamma_a^T\psi_A\gamma_a^T\psi_A)$
$+\frac{1}{2(m-2)}(1-\frac{1}{2}\pi(m-1)\beta(m))(\frac{m-3}{1-m}fL_{aa}+f_{;m})\gamma_m\psi_A\}\mathrm{d}y$.

As the interior integrands $a_1^{\eta,M}$ and $a_3^{\eta,M}$ were determined previously by Branson and Gilkey [11], we shall concentrate upon determining the boundary integrands.

Here is a brief outline to the paper. In section 2, we derive some basic functorial properties of these invariants. One of the peculiarities of using the 'functorial approach' is that it is necessary to work in a very general context and then specialize subsequently. To employ this method, we will have to work with operators which are not self-adjoint despite the fact that the examples which arise in practice are usually self-adjoint. In section 3, we express the invariants $a_n^{\eta,\partial M}$ in terms of a Weyl basis with certain undetermined coefficients and begin the evaluation of these coefficients. We complete the proof of theorem 1.2 in sections 4 and 5 by completing the determination of the coefficients.

2. Functorial properties

We refer to [22, 23] for the proof of the following result which describes the adjoint structures:

Lemma 2.1. *Let P^* be the formal adjoint of P, let A^* be the formal adjoint of A, and let $A^\# := \gamma_m A^* \gamma_m$.*

(i) The operator $A^\#$ defines the adjoint boundary condition for P^*.
(ii) We have $\psi_{P^*} = \psi_P^*$, $\psi_{A^*} = \psi_A^*$, and $\psi_{A^\#} = \gamma_m \psi_A^* \gamma_m + L_{aa}\mathrm{Id}$.
(iii) If ψ_P is self-adjoint, and if $\psi_A = \gamma_m \psi_A^* \gamma_m + L_{aa}\mathrm{Id}$, then P_A is self-adjoint on $L^2(V)$.

The next observation follows from work of Grubb and Seeley [26].

Lemma 2.2. *Let $n < m$. Assume that the metric on M is product near the boundary, that P_A is self-adjoint, that A is self-adjoint and that the coefficients of P and of A are independent of the normal variable near the boundary. Let F be an endomorphism of V whose coefficients are independent of the normal variable near the boundary.*

(i) If n is even, then $a_n^{\zeta,\partial M}(F, P, A) = -\frac{1}{2(m-n)\Gamma(\frac{1}{2})} a_{n-1}^{\eta}(F, A)$.
(ii) If n is odd, then $a_n^{\zeta,\partial M}(F, P, A) = \frac{1}{4}(\beta(m-n+1) - 1) a_{n-1}^{\zeta}(F, A)$.

Taking the adjoint yields yet another useful property.

Lemma 2.3. *Let $n < m$. Let (P, A) be real operators on a real bundle V. Suppose V is equipped with a fibre metric. Let P^* be the formal adjoint of P and let F^* be the adjoint of F. Set $A^\# = \gamma_m A^* \gamma_m$. Then $a_n^{\eta,\partial M}(F, P, A) = a_n^{\eta,\partial M}(F^*, P^*, A^\#)$.*

Proof. As we are in the real setting, taking the complex conjugate plays no role. Consequently

$$\mathrm{Tr}_{L^2}\{F P \,\mathrm{e}^{-tP_A^2}\} = \mathrm{Tr}_{L^2}\{F^* P^* \,\mathrm{e}^{-t(P_A^*)^2}\}.$$

The lemma follows by equating powers of t in the asymptotic expansions and by using lemma 2.1 to see that $A^\#$ defines the adjoint boundary condition. □

There is a useful relation between the ζ and the η invariants.

Lemma 2.4. *Let $F \in C^\infty(\mathrm{end}(V))$. Let (A, P) be as above and let $n < m$.*

(i) Let $P_\varepsilon := P + \varepsilon F$. Then
- *(a) $\partial_\varepsilon a_n^\eta(1, P_\varepsilon, A) = (n-m) a_{n-1}^\zeta(F, P_\varepsilon, A)$.*
- *(b) $\partial_\varepsilon a_n^\zeta(1, P_\varepsilon, A) = -2 a_{n-1}^\eta(F, P_\varepsilon, A)$.*

(ii) Let $P_\varepsilon := P + \varepsilon\,\mathrm{Id}$. Then
- *(a) $\partial_\varepsilon a_n^\eta(F, P_\varepsilon, A) = (n-m) a_{n-1}^\zeta(F, P_\varepsilon, A)$.*
- *(b) $\partial_\varepsilon a_n^\zeta(F, P_\varepsilon, A) = -2 a_{n-1}^\eta(F, P_\varepsilon, A)$.*

(iii) Let $P_\varepsilon := \mathrm{e}^{-\varepsilon f} P$ where f is a smooth scalar function vanishing on ∂M. Then $\partial_\varepsilon a_n^\eta(1, P_\varepsilon, A) = (m-n) a_n^\eta(f, P_\varepsilon, A)$.

Proof. To prove assertion (1), let $P_\varepsilon := P + \varepsilon F$. We compute

$$\sum_n \partial_\varepsilon a_n^\eta(1, P_\varepsilon, A) t^{(n-m-1)/2} \sim \partial_\varepsilon \operatorname{Tr}\{P_\varepsilon \,\mathrm{e}^{-tP_{\varepsilon,A}^2}\} = \operatorname{Tr}\{F(\mathrm{Id} - 2tP_\varepsilon^2)\,\mathrm{e}^{-tP_{\varepsilon,A}^2}\}$$

$$= (1 + 2t\partial_t)\operatorname{Tr}\{F\,\mathrm{e}^{-tP_{\varepsilon,A}^2}\} \sim (1 + 2t\partial_t)\sum_k a_k^\zeta(F, P_\varepsilon, A) t^{(k-m)/2}$$

$$= \sum_k (1 + k - m) a_k^\zeta(F, P_\varepsilon, A) t^{(k-m)/2}.$$

Setting $k = n - 1$ and equating terms in the asymptotic expansions establishes assertion ($1a$). Similarly, we compute

$$\sum_n \partial_\varepsilon a_n^\zeta(1, P_\varepsilon, A) t^{(n-m)/2} \sim \partial_\varepsilon \operatorname{Tr}\{\mathrm{e}^{-tP_{\varepsilon,A}^2}\}$$

$$= -2t \operatorname{Tr}\{FP_\varepsilon\,\mathrm{e}^{-tP_{\varepsilon,A}^2}\} \sim \sum_k -2a_k^\eta(F, P_\varepsilon, A) t^{(k-m+1)/2}.$$

Again, equating coefficients in the associated asymptotic expansions yields assertion (1b); the proof of assertion (2) is similar and is therefore omitted. To prove assertion (3), we compute

$$\sum_n \partial_\varepsilon a_n^\eta(1, P_\varepsilon, A) t^{(n-m-1)/2} \sim \partial_\varepsilon \operatorname{Tr}\{P_\varepsilon\,\mathrm{e}^{-tP_{\varepsilon,A}^2}\}$$

$$= -\operatorname{Tr}\{f(P_\varepsilon - 2tP_\varepsilon^3)\,\mathrm{e}^{-tP_{\varepsilon,A}^2}\} = -(1 + 2t\partial_t)\operatorname{Tr}\{fP_\varepsilon\,\mathrm{e}^{-tP_{\varepsilon,A}^2}\}$$

$$= -\sum_n (1 + (n - m - 1)) a_n^\eta(f, P_\varepsilon, A) t^{(n-m-1)/2}.$$

Assertion (3) now follows by equating coefficients in the asymptotic expansions. □

We will need the following lemma to apply lemma 2.4. It involves a formula for endomorphism valued smearing functions which is related to the product case and which generalizes the formula of theorem 1.1 (3).

Lemma 2.5. *Assume that the metric on M is product near the boundary, that P_A is self-adjoint, that A is self-adjoint, and that the coefficients of P and A are independent of the normal variable near the boundary. Let F be an endomorphism of V whose coefficients are independent of the normal variable near the boundary. If $m \geqslant 3$, then*

$$a_2^\zeta(F, P, A) = (4\pi)^{-m/2}\int_M \operatorname{Tr}\left\{F\left(\frac{1}{6}\tau + E\right)\right\}\mathrm{d}x$$

$$- \frac{1}{2(m-2)}(4\pi)^{-m/2}\int_{\partial M}\operatorname{Tr}\{(3-m)F\psi_A + F\gamma_a^T\psi_A\gamma_a^T\}\,\mathrm{d}y.$$

Remark 2.6. To ensure that P_A is self-adjoint, we impose the relations of lemma 2.1 (3). Since $L_{aa} = 0$ by assumption, this means that $\psi_A = \gamma_m\psi_A\gamma_m$ and hence $\operatorname{Tr}\{\psi_A\} = 0$. Thus $a_2^{\zeta,\partial M}(\mathrm{Id}, P, A) = 0$; this is in agreement with theorem 1.1 (3).

Proof. We refer to [12] for the determination of the interior integrand. Let $N = \partial M$. We apply theorem 1.1 to the operator A on the closed manifold N to see

$$a_2^\zeta(1, A) = -\frac{1}{12}(4\pi)^{-(m-1)/2}\int_N \operatorname{Tr}\{\tau\mathrm{Id} + (12 - 6(m-1))\psi_A^2 + 6\psi_A\gamma_a^T\psi_A\gamma_a^T\}\,\mathrm{d}y.$$

We set $A_\varepsilon := A + \varepsilon F$. By lemma 2.4, with an appropriate dimension shift,

$$-2a_1^\eta(F, A) = \partial_\varepsilon|_{\varepsilon=0}\, a_2^\zeta(1, A_\varepsilon)$$

$$= -\frac{1}{6}(4\pi)^{-(m-1)/2}\int_N \operatorname{Tr}\{F[(18 - 6m)\psi_A + 6\gamma_a^T\psi_A\gamma_a^T]\}\,\mathrm{d}y.$$

Combining this result with lemma 2.2 (1) then yields

$$a_2^{\zeta,\partial M}(F,P,A) = -\frac{1}{2(m-2)\sqrt{\pi}} a_1^{\eta}(F,A)$$
$$= -\frac{1}{12(m-2)}(4\pi)^{-m/2}\int_{\partial M}\mathrm{Tr}\big\{(18-6m)F\psi_A + 6F\gamma_a^T\psi_A\gamma_a^T\big\}\,\mathrm{d}y. \qquad \square$$

3. A formula with universal coefficients

As $a_n^\eta(F,-P,A) = -a_n^\eta(F,P,A)$, the boundary contributions, which are homogeneous of weight $n-1$, must be odd functions of P. Consequently, they vanish for $n = 0, 1$; assertions (1) and (2) of theorem 1.2 now follow. Furthermore, we have:

Lemma 3.1. *There exist universal constants $c_i(m)$ so that*

(i) $a_2^{\eta,\partial M}(f,P,A) = (4\pi)^{-(m-1)/2}\int_{\partial M} f\,\mathrm{Tr}\big\{c_m^1\psi_P + c_m^2\gamma_m\psi_A\big\}\,\mathrm{d}y.$

(ii) $a_3^{\eta,\partial M}(f,P,A) = (4\pi)^{-m/2}\int_{\partial M}\mathrm{Tr}\big\{c_m^3 f\gamma_m\psi_P^2 + c_m^4 f\gamma_m\gamma_a^T\psi_P\gamma_a^T\psi_P + c_m^5 f\gamma_m\psi_A^2$
$+ c_m^6 f\psi_P\psi_A + c_m^7 f\gamma_m\psi_P\gamma_m\psi_A + c_m^8 f\gamma_a^T\psi_P\gamma_a^T\psi_A + c_m^9 f\gamma_a\psi_P\gamma_a\psi_A$
$+ c_m^{10} f\gamma_m\gamma_a^T\psi_A\gamma_a^T\psi_A + c_m^{11} f\psi_{P;m} + c_m^{12} fL_{aa}\psi_P + c_m^{13} f_{;m}\psi_P + c_m^{14} f\big(\gamma_a^T\psi_P\big)_{:a}$
$+ c_m^{15} fL_{aa}\gamma_m\psi_A + c_m^{16} f_{;m}\gamma_m\psi_A + c_m^{17} f(\gamma_a\psi_A)_{:a}\big\}\,\mathrm{d}y.$

Many invariants do not occur because the trace over an odd number of γ-matrices is zero. Furthermore, invariants of the form $W_{ab}\gamma_m\gamma_a\gamma_b$ and $W_{am}\gamma_a$ are omitted as their trace vanishes as well.

We begin our study of these coefficients by varying the compatible connection chosen.

Lemma 3.2. *We have the relations:*

(i) $c_m^3 = 0.$

(ii) $c_m^6 - c_m^7 + (m-1)c_m^8 + (m-1)c_m^9 = 0.$

(iii) $c_m^6 + c_m^7 + (m-3)c_m^8 - (m-3)c_m^9 + 2(m-3)c_m^4 = 0.$

(iv) $c_m^6 + c_m^7 - (m-3)c_m^8 + (m-3)c_m^9 + 2(m-3)c_m^{10} = 0.$

Proof. There always exist Hermitian connections so $\nabla\gamma = 0$, see for example [11]. There are, however, many such connections. If $\omega := \varrho_i\,\mathrm{e}^i$ is a purely imaginary 1 form, then $\tilde\nabla := \nabla - \omega\mathrm{Id}$ is again a Hermitian connection with $\tilde\nabla\gamma = 0$. One has

$$\tilde\psi_P = \psi_P + \varrho_i\gamma_i \qquad \text{and} \qquad \tilde\psi_A = \psi_A + \varrho_b\gamma_b^T.$$

Clearly a_n^η does not depend on the particular connection chosen. We exhibit the terms which are linear in ϱ and omit the remaining terms to derive the following equations from which the desired relations of the lemma will follow:

$$\mathrm{Tr}\big\{c_m^3\gamma_m\tilde\psi_P^2\big\} = -2c_m^3\varrho_m\mathrm{Tr}\{\psi_P\} + \cdots,$$

$$\mathrm{Tr}\big\{c_m^4\gamma_m\gamma_a^T\tilde\psi_P\gamma_a^T\tilde\psi_P\big\} = c_m^4\,\mathrm{Tr}\{-2(m-3)\gamma_m\varrho_b\gamma_b\psi_P\} + \cdots,$$

$$\mathrm{Tr}\big\{c_m^5\gamma_m\tilde\psi_A^2\big\} = 0 + \cdots,$$

$$\mathrm{Tr}\big\{c_m^6\tilde\psi_P\tilde\psi_A\big\} = c_m^6\,\mathrm{Tr}\big\{\varrho_m\gamma_m\psi_A + \varrho_b\gamma_b\psi_A + \psi_P\varrho_b\gamma_b^T\big\} + \cdots,$$

$$\mathrm{Tr}\big\{c_m^7\gamma_m\tilde\psi_P\gamma_m\tilde\psi_A\big\} = c_m^7\,\mathrm{Tr}\big\{-\varrho_m\gamma_m\psi_A + \varrho_b\gamma_b\psi_A + \psi_P\varrho_b\gamma_b^T\big\} + \cdots,$$

$$\mathrm{Tr}\{c_m^8\gamma_a^T\tilde\psi_P\gamma_a^T\tilde\psi_A\} = c_m^8\,\mathrm{Tr}\{(m-1)\varrho_m\gamma_m\psi_A - (m-3)\varrho_b\gamma_b\psi_A + (m-3)\psi_P\varrho_b\gamma_b^T\} + \cdots,$$
$$\mathrm{Tr}\{c_m^9\gamma_a\tilde\psi_P\gamma_a\tilde\psi_A\} = c_m^9\,\mathrm{Tr}\{(m-1)\varrho_m\gamma_m\psi_A + (m-3)\varrho_b\gamma_b\psi_A - (m-3)\psi_P\varrho_b\gamma_b^T\} + \cdots,$$
$$\mathrm{Tr}\{c_m^{10}\gamma_m\gamma_a^T\tilde\psi_A\gamma_a^T\tilde\psi_A\} = c_m^{10}\,\mathrm{Tr}\{2(m-3)\gamma_m\varrho_b\gamma_b^T\psi_A\} + \cdots.$$
□

We shift the spectrum of A to show

Lemma 3.3. *We have the relations:*

(i) $c_m^5 = 0$.
(ii) $c_m^6 = c_m^7$ *and* $c_m^8 = -c_m^9$.

Proof. If we replace A by $A + \varepsilon\mathrm{Id}$, then the boundary condition is unchanged for small values of ε. We set $\tilde\psi_A := \psi_A + \varepsilon\mathrm{Id}$, exhibit only the linear terms, and omit all terms which are not linear in ε to derive the following equations:

$$\mathrm{Tr}\{c_m^5\gamma_m\tilde\psi_A^2\} = 2c_m^5\,\mathrm{Tr}\{\gamma_m\varepsilon\psi_A\} + \cdots,$$
$$\mathrm{Tr}\{c_m^6\psi_P\tilde\psi_A\} = c_m^6\,\mathrm{Tr}\{\varepsilon\psi_P\} + \cdots,$$
$$\mathrm{Tr}\{c_m^7\gamma_m\psi_P\gamma_m\tilde\psi_A\} = -c_m^7\,\mathrm{Tr}\{\varepsilon\psi_P\} + \cdots,$$
$$\mathrm{Tr}\{c_m^8\gamma_a^T\psi_P\gamma_a^T\tilde\psi_A\} = c_m^8\,\mathrm{Tr}\{-(m-1)\varepsilon\psi_P\} + \cdots,$$
$$\mathrm{Tr}\{c_m^9\gamma_a\psi_P\gamma_a\tilde\psi_A\} = c_m^9\,\mathrm{Tr}\{-(m-1)\varepsilon\psi_P\} + \cdots,$$
$$\mathrm{Tr}\{c_m^{10}\gamma_m\gamma_a^T\tilde\psi_A\gamma_a^T\tilde\psi_A\} = 0 + \cdots.$$

Assertion (1) follows. Furthermore, we have

$$0 = c_m^6 - c_m^7 - (m-1)c_m^8 - (m-1)c_m^9.$$

Assertion (2) follows from this equation and from lemma 3.2 (2). □

Lemma 3.4. *We have* $c_m^{14} = 0$ *and* $c_m^{17} = 0$.

Proof. We work on the flat annulus $M := \mathbb{T}^{m-1} \times [0, 1]$. Let h_a and H_a be real smooth functions on M. We set

$$P = \gamma_i\partial_i^x + \varepsilon h_a\gamma_a^T \qquad \text{and} \qquad A = \gamma_a^T\partial_a^y + \varepsilon H_b\gamma_b.$$

Let $F = f\cdot\mathrm{Id}$ be scalar. The presence of the smearing function f ensures the boundary and interior integrals do not interact. Modulo terms which are $O(\varepsilon^2)$, one has

$$a_3^\eta(F, P, A) = -\varepsilon(4\pi)^{-m/2}\int_{\partial M}\mathrm{Tr}\big(f\{c_m^{14}h_{b:b} + c_m^{17}H_{b:b}\}\big)\,\mathrm{d}y + O(\varepsilon^2).$$

By lemma 2.1,

$$P^* = \gamma_i\partial_i^x - \varepsilon h_a\gamma_a^T, \qquad \psi_{P^*} = -\varepsilon h_a\gamma_a^T,$$
$$A^\# = \gamma_m A^*\gamma_m = \gamma_m\big(\gamma_a^T\partial_a^y - \varepsilon H_b\gamma_b\big)\gamma_m, \qquad \psi_{A^\#} = -\varepsilon H_b\gamma_b.$$

Consequently, there is a sign change

$$a_3^\eta(F, P^*, A^\#) = \varepsilon(4\pi)^{-m/2}\int_{\partial M}\mathrm{Tr}\big(f\{c_m^{14}h_{b:b} + c_m^{17}H_{b:b}\}\big)\mathrm{d}y + O(\varepsilon^2).$$

By lemma 2.3, $a_3^\eta(f\mathrm{Id}, P^*, A^\#) = a_3^\eta(f\mathrm{Id}, P, A)$; the lemma follows. □

We use conformal variations to show

Lemma 3.5. $c_m^{15} = \frac{m-3}{1-m}c_m^{16}$.

Proof. Let f be a smooth function with $f|_{\partial M} = 0$. Let $\mathrm{d}s^2(\varepsilon) = \mathrm{e}^{2\varepsilon f}\,\mathrm{d}s^2$ and let $P(\varepsilon) := \mathrm{e}^{-\varepsilon f} P$. Let ∇ be a unitary connection with $\nabla\gamma = 0$. Let $x = (x_1, \ldots, x_m)$ be a system of local coordinates on M. Expand $P = \gamma^\nu \nabla_{\partial_\nu} + \psi_P$ and use the metric to lower indices and define γ_ν. Define a smooth 1 parameter family of connections

$$\nabla(\varepsilon)_{\partial_\mu} := \nabla_{\partial_\mu} + \frac{\varepsilon}{2}\{f_{;\nu}\gamma^\nu\gamma_\mu + f_{;\mu}\}.$$

Results of [16] show $\nabla(\varepsilon)\gamma(\varepsilon) = 0$ and $\nabla(\varepsilon)$ is unitary. Furthermore,

$$\psi_P(\varepsilon) = \mathrm{e}^{-\varepsilon f}\left\{\psi_P - \frac{m-1}{2}\varepsilon f_{;i}\gamma_i\right\} \qquad \text{and} \qquad \psi_A(\varepsilon) = \psi_A.$$

We suppose $\psi_P = 0$. We study the term $\mathrm{Tr}\{f_{;m}\gamma_m\psi_A\}$ and compute:

$$\partial_\varepsilon|_{\varepsilon=0}\,\mathrm{Tr}\big\{c^6_m\psi_P\psi_A + c^7_m\gamma_m\psi_P\gamma_m\psi_A\big\} = -\frac{m-1}{2}\big(c^6_m - c^7_m\big)\,\mathrm{Tr}\{f_{;m}\gamma_m\psi_A\} = 0,$$

$$\partial_\varepsilon|_{\varepsilon=0}\,\mathrm{Tr}\big\{c^8_m\gamma_a^T\psi_P\gamma_a^T\psi_A + c^9_m\gamma_a\psi_P\gamma_a\psi_A\big\} = -\frac{(m-1)^2}{2}\big\{c^8_m + c^9_m\big\}\,\mathrm{Tr}\{f_{;m}\gamma_m\psi_A\} = 0,$$

$$\partial_\varepsilon|_{\varepsilon=0}L_{aa} = (1-m)f_{;m}.$$

We concentrate on the term $\mathrm{Tr}\{f_{;m}\gamma_m\psi_A\}$ and compute

$$\partial_\varepsilon|_{\varepsilon=0}\,a^\eta_3(1, P(\varepsilon), A) = (4\pi)^{-m/2}\int_{\partial M} c^{15}_m\,\mathrm{Tr}\{(1-m)f_{;m}\gamma_m\psi_A\}\,\mathrm{d}y$$
$$= (m-3)a^\eta_3(f, P(\varepsilon), A) = (4\pi)^{-m/2}\int_{\partial M} c^{16}_m\,\mathrm{Tr}\{(m-3)f_{;m}\gamma_m\psi_A\}\,\mathrm{d}y.$$

The lemma now follows. □

We study a variation of the form $P_\varepsilon := P + \varepsilon\mathrm{Id}$ to establish

Lemma 3.6.

(i) $c^1_m = \frac{2-m}{4}(\beta(m) - 1)$.
(ii) $c^{12}_m = \frac{3-m}{3}\big(1 - \frac{3}{4}\pi\beta(m)\big)$ *and* $c^{13}_m = \frac{(m-3)(m-1)}{2(m-2)}\big(1 - \frac{1}{2}\pi\beta(m)\big)$.

Proof. Let ψ_P be self-adjoint. Set $\psi_A := \frac{1}{2}L_{aa}\mathrm{Id}$; then $A^\# = A^* = A$ and P_A is self-adjoint. Let $P_\varepsilon := P + \varepsilon\mathrm{Id}$. By theorem 1.1 and lemma 2.4

$$\partial_\varepsilon|_{\varepsilon=0}\,a^{\eta,\partial M}_2(f, P_\varepsilon, A) = (4\pi)^{-(m-1)/2}\int_M c^1_m f\,\mathrm{Tr}\{\mathrm{Id}\}\,\mathrm{d}y = (2-m)a^{\zeta,\partial M}_1(f, P, A)$$
$$= (4\pi)^{-(m-1)/2}\frac{2-m}{4}(\beta(m) - 1)\int_{\partial M} f\,\mathrm{Tr}\{\mathrm{Id}\}\,\mathrm{d}y$$

Assertion (1) follows. To establish assertion (2), we compute

$$\mathrm{Tr}\big\{c^6_m\psi_P\psi_A + c^7_m\gamma_m\psi_P\gamma_m\psi_A\big\} = \frac{1}{2}\big(c^6_m - c^7_m\big)\,\mathrm{Tr}\{\psi_P L_{aa}\} = 0,$$

$$\mathrm{Tr}\big\{c^8_m\gamma_a^T\psi_P\gamma_a^T\psi_A + c^9_m\gamma_a\psi_P\gamma_a\psi_A\big\} = \frac{(1-m)}{2}\big(c^8_m + c^9_m\big)\,\mathrm{Tr}\{\psi_P L_{aa}\} = 0,$$

$$\partial_\varepsilon|_{\varepsilon=0}\,c^4_m\,\mathrm{Tr}\big\{\gamma_m\gamma_a^T\psi_P\gamma_a^T\psi_P\big\} = c^4_m\,\mathrm{Tr}\big(\gamma_m\gamma_a^T\gamma_a^T\psi_P + \gamma_a^T\gamma_m\gamma_a^T\psi_P\big) = 0.$$

Consequently again by theorem 1.1 and lemma 2.4 one has

$$\partial_\varepsilon|_{\varepsilon=0}\,a^{\eta,\partial M}_3(f, P_\varepsilon, A) = (4\pi)^{-m/2}\int_{\partial M}\mathrm{Tr}\big\{c^{13}_m f_{;m}\,\mathrm{Id} + c^{12}_m f L_{aa}\big\}\,\mathrm{d}y$$
$$= (3-m)a^{\zeta,\partial M}_2(f, P, A)$$

$$= (3-m)\int_{\partial M}\left\{\frac{1}{3}\left(1-\frac{3}{4}\pi\beta(m)\right)L_{aa}f - \frac{m-1}{2(m-2)}\left(1-\frac{1}{2}\pi\beta(m)\right)f_{;m}\right\}\mathrm{Tr}\{\mathrm{Id}\}\,\mathrm{d}y.$$

Assertion (2) follows. □

4. The variation $P_\varepsilon := P + \varepsilon F$

In this section, we will study $\partial_\varepsilon a_3^\eta(1, P_\varepsilon, A)$. There is a non-trivial interaction between the boundary and interior integrals that must be dealt with. Our basic identity is provided by lemma 2.4,

$$\partial_\varepsilon|_{\varepsilon=0}\, a_3^\eta(1, P_\varepsilon, A) = (3-m)a_2^\zeta(F, P, A). \tag{4a}$$

Let F be endomorphism valued. Then

$$\begin{aligned}\partial_\varepsilon|_{\varepsilon=0}\, a_3^{\eta,M}(1, P_\varepsilon, A) = -\frac{1}{12}(4\pi)^{-m/2}\int_M \mathrm{Tr}\{&[2(m-1)F_{;i} + 3(4-m)F\gamma_i\psi_P\\ &+3(4-m)F\psi_P\gamma_i + 3F\gamma_j\gamma_i\psi_P\gamma_j + 3F\gamma_j\psi_P\gamma_j\gamma_i]_{;i}\\ &+(3-m)[F\tau + 6F\gamma_i\gamma_j W_{ij} - 6F\psi_{P;i}\gamma_i - 6F_{;i}\gamma_i\psi_P + 3(4-m)F\psi_P\psi_P\\ &+3F\psi_P\gamma_i\psi_P\gamma_i + 3F\gamma_i\psi_P\gamma_i\psi_P + 3F\gamma_i\psi_P\psi_P\gamma_i]\}\,\mathrm{d}x.\end{aligned}$$

On the other hand, by lemma 2.5,

$$\begin{aligned}a_2^{\zeta,M}(F, P, A) = -\frac{1}{12}(4\pi)^{-m/2}\int_M \mathrm{Tr}\Big\{F\Big(&\tau + 6\gamma_i\gamma_j W_{ij} + 6\gamma_i\psi_{P;i} - 6\psi_{P;i}\gamma_i + 12\psi_P^2\\ &+3\psi_P\gamma_i\psi_P\gamma_i + 3\gamma_i\psi_P\psi_P\gamma_i + 3\gamma_i\psi_P\gamma_i\psi_P - 3m\psi_P^2\Big)\Big\}\,\mathrm{d}x.\end{aligned}$$

Consequently, we may integrate by parts to see

$$\begin{aligned}\partial_\varepsilon a_3^{\eta,M}(1, P_\varepsilon, A)|_{\varepsilon=0} - (3-m)a_2^{\zeta,M}(F, P, A) &= -\frac{1}{12}(4\pi)^{-m/2}\int_M \mathrm{Tr}\{[2(m-1)F_{;i}\\ &+3(4-m)F\gamma_i\psi_P + 3(4-m)F\psi_P\gamma_i + 3F\gamma_j\gamma_i\psi_P\gamma_j + 3\gamma_j\psi_P\gamma_j\gamma_i F]_{;i}\\ &-6(3-m)F_{;i}\gamma_i\psi_P - 6(3-m)F\gamma_i\psi_{P;i}\}\,\mathrm{d}x\\ &= \frac{1}{12}(4\pi)^{-m/2}\int_{\partial M}\mathrm{Tr}\{2(m-1)F_{;m} + 3(4-m)F\gamma_m\psi_P + 3(4-m)F\psi_P\gamma_m\\ &+3F\gamma_j\gamma_m\psi_P\gamma_j + 3F\gamma_j\psi_P\gamma_j\gamma_m - 6(3-m)F\gamma_m\psi_P\}\,\mathrm{d}y.\end{aligned} \tag{4b}$$

After setting $c_m^3 = c_m^5 = 0$, $c_m^7 = c_m^6$ and $c_m^9 = -c_m^8$, one has

$$\begin{aligned}\partial_\varepsilon a_3^{\eta,\partial M}(1, P_\varepsilon, A)|_{\varepsilon=0} = (4\pi)^{-m/2}\int_{\partial M}\mathrm{Tr}\Big\{&c_m^4 F\Big(\gamma_a^T\psi_P\gamma_m\gamma_a^T + \gamma_m\gamma_a^T\psi_P\gamma_a^T\Big)\\ &+c_m^6 F(\psi_A + \gamma_m\psi_A\gamma_m) + c_m^8 F\Big(\gamma_a^T\psi_A\gamma_a^T - \gamma_a\psi_A\gamma_a\Big) + c_m^{11}F_{;m} + c_m^{12}FL_{aa}\Big\}\,\mathrm{d}y.\end{aligned} \tag{4c}$$

There are several different settings where we know $a_2^{\zeta,\partial M}$. For the next two lemmas, to ensure that P_A is self-adjoint, we shall assume ψ_P and ψ_A are self adjoint and that $\psi_A = \gamma_m\psi_A\gamma_m + L_{aa}\mathrm{Id}$. We begin by applying theorem 1.1.

Lemma 4.1. *We have*

$$c_m^{11} = \frac{(m-3)(m-1)}{2(m-2)}\left(1-\frac{1}{2}\pi\beta(m)\right) - \frac{1}{6}(m-1).$$

Proof. We take $F = f \cdot \mathrm{Id}$ to be scalar and set $P_\varepsilon := P + \varepsilon F$. The terms involving $\mathrm{Tr}(\psi_A)$ and $\mathrm{Tr}(\gamma_m \psi_P)$ cancel and we have

$$0 = \partial_\varepsilon a_3^\eta(1, P_\varepsilon, A)|_{\varepsilon=0} - (3-m)a_2^\zeta(F, P, A)$$
$$= \mathrm{Tr}\{\mathrm{Id}\}(4\pi)^{-m/2} \int_{\partial M} \left\{ \left(\frac{1}{6}(m-1) + c_m^{11} \right) f_{;m} + c_m^{12} f L_{aa} \right.$$
$$\left. - \frac{3-m}{3}\left(1 - \frac{3}{4}\pi\beta(m)\right) L_{aa} f + \frac{(m-1)(3-m)}{2(m-2)}\left(1 - \frac{1}{2}\pi\beta(m)\right) f_{;m} \right\} \mathrm{d}y.$$

We equate the coefficients of $f L_{aa}$ to determine a value for c_m^{12} which agrees with that obtained in lemma 3.6. Equating the coefficients of $f_{;m}$ determines c_m^{11}. □

We apply lemma 2.5 to prove

Lemma 4.2. *We have the relations:*

(i) $c_m^6 = -\frac{(3-m)^2}{4(m-2)}$, *and* $c_m^8 = -\frac{3-m}{4(m-2)}$.

(ii) $c_m^4 = 0$, *and* $c_m^{10} = -2\frac{(3-m)}{4(m-2)}$.

Proof. We assume the structures are product near the boundary. We first study the terms $\mathrm{Tr}\{F\psi_A\}$ and $\mathrm{Tr}\{\gamma_a F \gamma_a \psi_A\}$. Since $L_{aa} = 0$, $\gamma_m \psi_A \gamma_m = \psi_A$. We compute using equations (4*b*) and (4*c*) that

$$c_m^6 \partial_\varepsilon|_{\varepsilon=0} \mathrm{Tr}\{\psi_P \psi_A + \gamma_m \psi_P \gamma_m \psi_A\} = 2c_m^6 \mathrm{Tr}\{F\psi_A\},$$
$$c_m^8 \partial_\varepsilon|_{\varepsilon=0} \mathrm{Tr}\left\{\gamma_a^T \psi_P \gamma_a^T \psi_A - \gamma_a \psi_P \gamma_a \psi_A\right\} = -2c_m^8 \mathrm{Tr}\{\gamma_a F \gamma_a \psi_A\}.$$

Thus lemma 2.5 yields

$$(4\pi)^{-m/2} \int_{\partial M} \mathrm{Tr}\left\{2c_m^6 F\psi_A - 2c_m^8 F\gamma_a \psi_A \gamma_a\right\} \mathrm{d}y + \cdots$$
$$= -\frac{3-m}{2(m-2)}(4\pi)^{-m/2} \int_{\partial M} \mathrm{Tr}\{F(3-m)\psi_A - F\gamma_a\psi_A\gamma_a\}\,\mathrm{d}y + \cdots. \qquad (4d)$$

To complete the proof of assertion (1), we must show equation (4*d*) yields two linearly independent relations. If we set $F = \psi_A = \sqrt{-1}\gamma_1$, then $\psi_A^* = \psi_A$, $\gamma_m \psi_A \gamma_m = \psi_A$, and

$$\mathrm{Tr}(F\psi_A) = \mathrm{Tr}\{\mathrm{Id}\} \qquad \text{and} \qquad \mathrm{Tr}(F\gamma_a\psi_A\gamma_a) = (m-3)\,\mathrm{Tr}\{\mathrm{Id}\}.$$

If we set $F = \psi_A = \gamma_1\gamma_2\gamma_3$, then $\psi_A^* = \psi_A$, $\gamma_m \psi_A \gamma_m = A$, and

$$\mathrm{Tr}(F\psi_A) = \mathrm{Tr}\{\mathrm{Id}\} \qquad \text{and} \qquad \mathrm{Tr}(F\gamma_a\psi_a\gamma_a) = (m-7)\,\mathrm{Tr}\{\mathrm{Id}\}.$$

Assertion (1) follows. Assertion (1) and lemma 3.2 imply assertion (2). □

5. A special case calculation on the ball

In order to find the remaining unknown coefficients c_m^2 and c_m^{16} we evaluate the leading coefficients in the asymptotic of the eta invariant for an example on the ball. We first describe the setting considered.

Let $r \in [0, 1]$ be the radial normal coordinate and $\mathrm{d}\Sigma^2$ the usual metric on the unit sphere S^{m-1}. Then the standard metric on the ball is $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\Sigma^2$. The inward unit normal on the boundary is $-\partial_r$. For this metric, the only nonvanishing components of the Christoffel symbols are

$$\Gamma_{abc} = \frac{1}{r}\tilde{\Gamma}_{abc} \qquad \text{and} \qquad \Gamma_{abm} = \frac{1}{r}\delta_{ab};$$

the second fundamental form is given by $L_{ab} = \delta_{ab}$. We will use $\tilde{\Gamma}_{abc}$ to refer to the Christoffel symbols associated with the metric $\mathrm{d}\Sigma^2$ on the sphere S^{m-1}. We will consider the Dirac operator $P = \gamma^\nu \partial_\nu$ on the ball; we take the flat connection ∇ and set $\psi_P = 0$. We suppose m even (there is a corresponding decomposition for m odd) and use the following representation of the γ-matrices,

$$\gamma_{a(m)} = \begin{pmatrix} 0 & \sqrt{-1}\cdot\gamma_{a(m-1)} \\ -\sqrt{-1}\cdot\gamma_{a(m-1)} & 0 \end{pmatrix}$$

and

$$\gamma_{m(m)} = \begin{pmatrix} 0 & \sqrt{-1}\cdot 1_{m-1} \\ \sqrt{-1}\cdot 1_{m-1} & 0 \end{pmatrix}.$$

We stress that the matrices $\gamma_{j(m)}$ are the γ-matrices projected along some vielbein system e_j. We decompose $\nabla_j = e_j + \omega_j$ where $\omega_j = \frac{1}{4}\Gamma_{jkl}\gamma_{k(m)}\gamma_{l(m)}$ is the connection-1 form of the spin connection. If $\tilde{\nabla}$ denotes the connection on the sphere, we have

$$\nabla_a = \frac{1}{r}\left(\begin{pmatrix} \tilde{\nabla}_a & 0 \\ 0 & \tilde{\nabla}_a \end{pmatrix} + \frac{1}{2}\gamma^T_{a(m)}\right).$$

This allows us to decompose the Dirac operator on the ball into a radial part and a part living on the sphere. In detail, if $\tilde{P}$ is the Dirac operator on the sphere, we have

$$P = \left(\frac{\partial}{\partial x_m} - \frac{m-1}{2r}\right)\gamma_{m(m)} + \frac{1}{r}\begin{pmatrix} 0 & \sqrt{-1}\tilde{P} \\ -\sqrt{-1}\tilde{P} & 0 \end{pmatrix}.$$

Let d_s be the dimension of the spin bundle on the disk; $d_s = 2^{m/2}$ if m is even. The spinor modes $\mathcal{Z}_\pm^{(n)}$ on the sphere are discussed in [13]. We have

$$\tilde{P}\mathcal{Z}_\pm^{(n)}(\Omega) = \pm\left(n + \frac{m-1}{2}\right)\mathcal{Z}_\pm^{(n)}(\Omega) \qquad \text{for} \quad n = 0, 1, \ldots;$$

$$d_n(m) := \dim \mathcal{Z}_\pm^{(n)}(\Omega) = \frac{1}{2}d_s\binom{m+n-2}{n}.$$

Let $J_\nu(z)$ be the Bessel functions. These satisfy the differential equation [24]

$$\frac{\mathrm{d}^2 J_\nu(z)}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}J_\nu(z)}{\mathrm{d}z} + \left(1 - \frac{\nu^2}{z^2}\right)J_\nu(z) = 0.$$

Let $P\varphi_\pm = \pm\mu\varphi_\pm$ be an eigenfunction of P. Modulo a suitable radial normalizing constant C, we may express

$$\varphi_\pm^{(+)} = \frac{C}{r^{(m-2)/2}}\begin{pmatrix} \sqrt{-1}J_{n+m/2}(\mu r)Z_+^{(n)}(\Omega) \\ \pm J_{n+m/2-1}(\mu r)Z_+^{(n)}(\Omega) \end{pmatrix}, \tag{5a}$$

and

$$\varphi_\pm^{(-)} = \frac{C}{r^{(m-2)/2}}\begin{pmatrix} \pm J_{n+m/2-1}(\mu r)Z_-^{(n)}(\Omega) \\ \sqrt{-1}J_{n+m/2}(\mu r)Z_-^{(n)}(\Omega) \end{pmatrix}. \tag{5b}$$

We next impose the boundary conditions. We choose for $\epsilon \in \mathbb{R}$ the boundary endomorphism

$$\psi_A = \epsilon\gamma_{m(m)} + \tfrac{1}{2}L_{aa}\,\mathrm{Id} \tag{5c}$$

such that

$$\psi_A = \gamma_{m(m)}\psi_A^*\gamma_{m(m)} + L_{aa}\,\mathrm{Id}.$$

This guarantees that P_A is self-adjoint (see lemma 2.1, assertions (2) and (3)). For this setting the general form of the leading coefficients for the eta invariant are obtained from lemmas 3.1 and 3.5. Noting that the volume of the $(m-1)$-dimensional sphere is $2\pi^{m/2}/\Gamma(m/2)$, and that

$$\mathrm{Tr}\{\gamma_m \psi_A\} = -\epsilon d_s,$$

one finds

$$a_2^\eta(1, P, A) = -c_m^2 \frac{\epsilon d_s \sqrt{\pi}}{2^{m-2}\Gamma\left(\frac{m}{2}\right)}, \tag{5d}$$

$$a_3^\eta(1, P, A) = c_m^{16} \frac{(m-3)\epsilon d_s}{2^{m-1}\Gamma\left(\frac{m}{2}\right)}. \tag{5e}$$

Thus finding explicit answers for this example will allow us to determine c_m^2 and c_m^{16}. We proceed towards this goal.

For the ψ_A given in (5c) the boundary operator A is given by

$$A = \begin{pmatrix} -\tilde{P} & i\epsilon \\ i\epsilon & \tilde{P} \end{pmatrix}.$$

We need to find the spectral projection on those eigenspinors of A whose eigenvalues have a positive real part. The endomorphism ψ_A chosen allows us to obtain closed forms for all eigenvalues $\pm\mu_n$ and eigenspinors $\left(\alpha_1^\pm, \alpha_2^\pm\right)$ defined by the differential equation

$$A\begin{pmatrix} \alpha_1^\pm \\ \alpha_2^\pm \end{pmatrix} = \begin{pmatrix} -\tilde{P} & i\epsilon \\ i\epsilon & \tilde{P} \end{pmatrix}\begin{pmatrix} \alpha_1^\pm \\ \alpha_2^\pm \end{pmatrix} = \pm\mu_n \begin{pmatrix} \alpha_1^\pm \\ \alpha_2^\pm \end{pmatrix}.$$

Let

$$\lambda_n = n + \tfrac{1}{2}(m-1)$$

be the eigenvalues associated with $\epsilon = 0$ [23]. One can then show that

$$\mu_n = \sqrt{\lambda_n^2 - \epsilon^2}$$

and

$$\begin{pmatrix} \alpha_1^+ \\ \alpha_2^+ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{-1}\epsilon}{2\lambda_n} Z_+^{(n)} + Z_-^{(n)} \\ \frac{1}{2\lambda_n}\left(\sqrt{\lambda_n^2 - \epsilon^2} + \lambda_n\right) Z_+^{(n)} + \frac{1}{\sqrt{-1}\epsilon}\left(\sqrt{\lambda_n^2 - \epsilon^2} - \lambda_n\right) Z_-^{(n)} \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1^- \\ \alpha_2^- \end{pmatrix} = \begin{pmatrix} Z_+^{(n)} - \frac{\sqrt{-1}\epsilon}{2\lambda_n} Z_-^{(n)} \\ -\frac{1}{\sqrt{-1}\epsilon}\left(\sqrt{\lambda_n^2 - \epsilon^2} - \lambda_n\right) Z_+^{(n)} + \frac{1}{2\lambda_n}\left(\sqrt{\lambda_n^2 - \epsilon^2} + \lambda_n\right) Z_-^{(n)} \end{pmatrix}.$$

We choose $\epsilon < (m-1)/2$ such that all eigenvalues μ_n are real. The solutions are normalized such that in the limit $\epsilon \to 0$ they reduce to the previously determined solutions in [23].

We want to suppress the projection on the positive spectrum of A. Using the solutions given in equations (5a) and (5b) this is easily accomplished. Projecting $\varphi_\pm^{(+)}$ onto the positive spectrum of A gives the implicit eigenvalue condition

$$J_{\lambda_n - \frac{1}{2}}(\mu) \mp \frac{\epsilon}{\sqrt{\lambda_n^2 - \epsilon^2} + \lambda_n} J_{\lambda_n + \frac{1}{2}}(\mu) = 0,$$

whereas projecting $\varphi_\pm^{(-)}$ produces

$$J_{\lambda_n - \frac{1}{2}}(\mu) \pm \frac{1}{\epsilon}\left(\sqrt{\lambda_n^2 - \epsilon^2} - \lambda_n\right) J_{\lambda_n + \frac{1}{2}}(\mu) = 0.$$

Combining the equations for the positive eigenvalues of P_A, we have the condition

$$\left(J_{\lambda_n-\frac{1}{2}}(\mu) - \frac{\epsilon}{\sqrt{\lambda_n^2-\epsilon^2}+\lambda_n} J_{\lambda_n+\frac{1}{2}}(\mu)\right)\left(J_{\lambda_n-\frac{1}{2}}(\mu) + \frac{1}{\epsilon}\left(\sqrt{\lambda_n^2-\epsilon^2}-\lambda_n\right) J_{\lambda_n+\frac{1}{2}}(\mu)\right) = 0.$$

For the present purpose it will be sufficient to find the unknown multipliers c_m^2 and c_m^{16} multiplying a linear term in ψ_A. Therefore we only need to pick up linear terms in ϵ and we will consider only terms up to the order ϵ explicitly. Having that in mind we write the implicit eigenvalue condition for positive eigenvalues instead as

$$J_{\lambda_n-\frac{1}{2}}(\mu)\left(J_{\lambda_n-\frac{1}{2}}(\mu) - \frac{\epsilon}{\lambda_n} J_{\lambda_n+\frac{1}{2}}(\mu)\right) + \mathcal{O}(\epsilon^2) = 0. \tag{5f}$$

To simplify the notation, set

$$p = \lambda_n - \tfrac{1}{2} \qquad \text{and} \qquad d_n(m) = d_p(m).$$

Furthermore, we use the recursion for Bessel functions, see [24],

$$z\frac{\mathrm{d}}{\mathrm{d}z} J_p(z) - pJ_p(z) = -zJ_{p+1}(z),$$

to rewrite (5f) such that only the index p appears,

$$J_p(\mu)\left(J_p(\mu)\left[1 - \frac{\epsilon p}{\mu(p+1/2)}\right] + \frac{\epsilon}{p+1/2} J_p'(\mu)\right) + \mathcal{O}(\epsilon^2) = 0. \tag{5g}$$

Proceeding similarly with the negative eigenvalues of P_A the outcome is

$$J_p(\mu)\left(J_p(\mu)\left[1 + \frac{\epsilon p}{\mu(p+1/2)}\right] - \frac{\epsilon}{p+1/2} J_p'(\mu)\right) + \mathcal{O}(\epsilon^2) = 0. \tag{5h}$$

Using Cauchy's residue theorem these equations allow us to rewrite the eta function

$$\eta(s; P, A) = \sum_\mu (\mathrm{sign}(\mu))|\mu|^{-s}$$

in terms of a contour integral, a technique recently described in detail in [8–10, 29]. The coefficients in the asymptotic expansion (1c) are then determined by evaluating residues of η according to [21]

$$\mathrm{Res}\ \eta(m-n; P, A) = \frac{2a_n^\eta(1, P, A)}{\Gamma\left(\frac{m-n+1}{2}\right)}. \tag{5i}$$

We will need the residues at $s = m-2$ and $s = m-3$ in order to determine the coefficients a_2^η and a_3^η.

Neglecting systematically the higher order terms in ϵ, we use a suitable counterclockwise contour C enclosing all the solutions of the equations (5g) and (5h) to write the eta function as (from now on it will be understood that this is the eta function up to the order ϵ)

$$\begin{aligned}\eta(s; P, A) = {} & \sum_p d_p(m) \int_C \frac{\mathrm{d}k}{2\pi i} k^{-s} \frac{\partial}{\partial k} \\ & \times \left\{ \ln\left[J_p(k)\left(J_p(k)\left[1 - \frac{\epsilon p}{k(p+1/2)}\right] + \frac{\epsilon}{p+1/2} J_p'(k)\right)\right]\right. \\ & \left. - \ln\left[J_p(k)\left(J_p(k)\left[1 + \frac{\epsilon p}{k(p+1/2)}\right] - \frac{\epsilon}{p+1/2} J_p'(k)\right)\right]\right\} \\ = {} & \sum_p d_p(m) \int_C \frac{\mathrm{d}k}{2\pi i} k^{-s} \frac{\partial}{\partial k} \left\{ \ln\left[1 - \frac{\epsilon p}{k(p+1/2)} + \frac{\epsilon}{p+1/2} \frac{J_p'(k)}{J_p(k)}\right]\right.\end{aligned}$$

$$- \ln\left[1 + \frac{\epsilon p}{k(p+1/2)} - \frac{\epsilon}{p+1/2}\frac{J_p'(k)}{J_p(k)}\right]\Bigg\}$$

$$= \sum_p d_p(m) p^{-s} \int_C \frac{\mathrm{d}z}{2\pi i} z^{-s} \frac{\partial}{\partial z} \left\{ \ln\left[1 - \frac{\epsilon p}{zp(p+1/2)} + \frac{\epsilon}{p+1/2}\frac{J_p'(zp)}{J_p(zp)}\right] \right.$$

$$\left. - \ln\left[1 + \frac{\epsilon p}{zp(p+1/2)} - \frac{\epsilon}{p+1/2}\frac{J_p'(zp)}{J_p(zp)}\right]\right\}.$$

In the last equation we substituted $k = zp$ in order to allow later on for a straightforward application of the formulae for the uniform asymptotic expansion of the Bessel functions. Again, expanding up to the order ϵ term, we write instead

$$\eta(s; P, A) = 2\epsilon \sum_p d_p(m) p^{-s} \int_C \frac{\mathrm{d}z}{2\pi i} z^{-s} \frac{\partial}{\partial z} \left\{ -\frac{1}{z(p+1/2)} + \frac{1}{p+1/2}\frac{J_p'(zp)}{J_p(zp)} \right\}.$$

The next step in the procedure is to shift the contour towards the imaginary axis, turning the Bessel function J_p into the Bessel function I_p. In detail, we find

$$\eta(s; P, A) = -\frac{2\epsilon}{\pi} \cos\left(\frac{\pi s}{2}\right) \sum_p d_p(m) p^{-s} (p+1/2)^{-1} \int_0^\infty \mathrm{d}z\, z^{-s} \frac{\mathrm{d}}{\mathrm{d}z} \left\{ \frac{1}{z} - \frac{I_p'(zp)}{I_p(zp)} \right\}.$$

The residues of the eta function are completely determined by the asymptotic behaviour of the Bessel functions (see [29] for details). Therefore we need to introduce some additional notation dealing with the uniform asymptotic expansion of the Bessel function $I_p(k)$. For $p \to \infty$ with $z = k/p$ fixed, we make use of the uniform asymptotic expansion of the Bessel function $I_p(zp)$ and the derivative $I_p'(zp)$. In detail, the relevant results are [1]

$$I_p(zp) \sim \frac{1}{\sqrt{2\pi p}} \frac{\mathrm{e}^{p\eta}}{(1+z^2)^{1/4}} \left[1 + \sum_{l=1}^\infty \frac{u_l(t)}{p^l}\right],$$

$$I_p'(zp) \sim \frac{1}{\sqrt{2\pi p}} \frac{\mathrm{e}^{p\eta}(1+z^2)^{1/4}}{z} \left[1 + \sum_{l=1}^\infty \frac{v_l(t)}{p^l}\right], \tag{5j}$$

where

$$t = 1/\sqrt{1+z^2} \qquad \text{and} \qquad \eta = \sqrt{1+z^2} + \ln\left[z/(1+\sqrt{1+z^2})\right]. \tag{5k}$$

Let $u_0(t) = 1$. We use the recursion relationship given in [1] to determine the polynomials $u_l(t)$ and $v_l(t)$ which appear in equations (5j) and (5k),

$$u_{l+1}(t) = \frac{1}{2} t^2 (1-t^2) u_l'(t) + \frac{1}{8} \int_0^t \mathrm{d}\tau (1 - 5\tau^2) u_l(\tau),$$

$$v_l(t) = u_l(t) + t(t^2 - 1)\left[\frac{1}{2} u_{l-1}(t) + t u_{l-1}'(t)\right].$$

In particular we have

$$u_1(t) = \tfrac{1}{8} t - \tfrac{5}{24} t^3, \qquad v_1(t) = -\tfrac{3}{8} t + \tfrac{7}{24} t^3.$$

The required leading two contributions from the asymptotic expansion are then given by

$$B_0(s; P, A) = -\frac{2\epsilon}{\pi} \cos\left(\frac{\pi s}{2}\right) \sum_p d_p(m) p^{-s} (p+1/2)^{-1} \int_0^\infty \mathrm{d}z\, z^{-s} \frac{\mathrm{d}}{\mathrm{d}z} \left\{ \frac{1}{z}\left(1 - \sqrt{1+z^2}\right) \right\},$$

$$B_{-1}(s; P, A) = \frac{2\epsilon}{\pi} \cos\left(\frac{\pi s}{2}\right) \sum_p d_p(m) p^{-s} (p+1/2)^{-1}$$

$$\times \int_0^\infty \mathrm{d}z\, z^{-s} \frac{\mathrm{d}}{\mathrm{d}z} \left\{ \frac{\sqrt{1+z^2}}{z} \left(\frac{1}{p} [v_1(t) - u_1(t)] \right) \right\}$$
$$= -\frac{\epsilon}{\pi} \cos\left(\frac{\pi s}{2}\right) \sum_p d_p(m) p^{-s-1} (p+1/2)^{-1} \int_0^\infty \mathrm{d}z\, z^{-s} \frac{\mathrm{d}}{\mathrm{d}z} \frac{z}{1+z^2}.$$

The integrals can be evaluated with the help of the beta function (see [24]). Using

$$\Gamma\left(-\frac{1+s}{2}\right) = -\frac{\pi}{\cos\left(\frac{\pi s}{2}\right) \Gamma\left(\frac{3+s}{2}\right)}$$

the answers obtained are

$$B_0(s; P_A) = -\frac{\epsilon}{\pi} \cos\left(\frac{\pi s}{2}\right) \frac{\Gamma\left(-\frac{s+1}{2}\right) \Gamma\left(1+\frac{s}{2}\right)}{\sqrt{\pi}} \sum_p d_p(m) p^{-s} (p+1/2)^{-1}$$
$$= \epsilon \frac{\Gamma\left(1+\frac{s}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{3+s}{2}\right)} \sum_p d_p(m) p^{-s} (p+1/2)^{-1}$$
$$B_{-1}(s; P_A) = -\frac{s\epsilon}{2} \sum_p d_p(m) p^{-s-1} (p+1/2)^{-1}.$$

The remaining summations are related to the spectrum on the sphere. Let $d := m - 1$. We define the base zeta-function ζ_{S^d} and the Barnes zeta-function $\zeta_{\mathcal{B}}$ [6],

$$\zeta_{S^d}(s) = \sum_{n=0}^\infty d_n(m) p^{-2s} \qquad \text{and} \qquad \zeta_{\mathcal{B}}(s,a) = \sum_{n=0}^\infty d_n(m)(n+a)^{-s}.$$

We then have the relation

$$\zeta_{S^d}(s) = \frac{1}{2} d_s \zeta_{\mathcal{B}}\left(2s, \frac{m}{2} - 1\right).$$

Using the Barnes zeta-function, up to terms that are irrelevant for the present purpose because their residues are located to the left of $s = m - 3$, we find

$$B_0(s; P, A) = \frac{1}{2} d_s \epsilon \frac{\Gamma\left(1+\frac{s}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{3+s}{2}\right)} \left\{ \zeta_{\mathcal{B}}\left(s+1, \frac{m}{2} - 1\right) - \frac{1}{2} \zeta_{\mathcal{B}}\left(s+2, \frac{m}{2} - 1\right) + \cdots \right\},$$
$$B_{-1}(s; P, A) = -\frac{1}{4} s\epsilon d_s \left\{ \zeta_{\mathcal{B}}\left(s+2, \frac{m}{2} - 1\right) + \cdots \right\}.$$

This reduces the analysis of the eta function on the ball to the analysis of $\zeta_{\mathcal{B}}(s,a)$. To compute the relevant residues, we first express $\zeta_{\mathcal{B}}(s,a)$ as a contour integral. Let $\mathcal{C}$ be the Hankel contour.

$$\zeta_{\mathcal{B}}(s,a) = \sum_{n=0}^\infty \binom{d+n-1}{n} (n+a)^{-s} = \sum_{\vec{m} \in \mathbb{N}_0^d} (a + m_1 + \cdots + m_d)^{-s}$$
$$= \frac{\Gamma(1-s)}{2\pi} \int_{\mathcal{C}} \mathrm{d}t\, (-t)^{s-1} \frac{\mathrm{e}^{-at}}{(1-\mathrm{e}^{-t})^d}.$$

The residues of $\zeta_{\mathcal{B}}(s,a)$ are intimately connected with the generalized Bernoulli polynomials [38],

$$\frac{\mathrm{e}^{-at}}{(1-\mathrm{e}^{-t})^d} = (-1)^d \sum_{n=0}^\infty \frac{(-t)^{n-d}}{n!} B_n^{(d)}(a). \tag{5l}$$

We use the residue theorem to see that

$$\mathrm{Res}_{s=z}\zeta_{\mathcal{B}}(s,a) = \frac{(-1)^{d+z}}{(z-1)!(d-z)!}B^{(d)}_{d-z}(a), \tag{5m}$$

for $z = 1, \ldots, d$. The required leading poles are

$$\mathrm{Res}_{s=d}\zeta_{\mathcal{B}}(s,a) = \frac{1}{(d-1)!}, \qquad \mathrm{Res}_{s=d-1}\zeta_{\mathcal{B}}(s,a) = \frac{d-2a}{2(d-2)!}.$$

This shows

$$\mathrm{Res}\, B_0(d-1;P,A) = \frac{1}{2}d_s\epsilon\frac{\Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m+1}{2}\right)(m-2)!},$$

$$\mathrm{Res}\, B_0(d-2;P,A) = \frac{1}{4}d_s\epsilon\frac{\Gamma\left(\frac{m-1}{2}\right)(m-3)}{\sqrt{\pi}\Gamma\left(\frac{m}{2}\right)(m-2)!},$$

$$\mathrm{Res}\, B_{-1}(d-2;P,A) = -\frac{1}{4}d_s\epsilon\frac{(m-3)}{(m-2)!},$$

and these are all the terms contributing to the residues of η at $s = d-1$ and $s = d-2$. Comparing with (5*d*) and (5*e*), after suitable rearrangements of the Γ-function [24], we use the doubling formula

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right),$$

and $\Gamma(x+1) = x\Gamma(x)$, we read off

$$c_m^2 = -\frac{1}{4}\beta(m), \qquad c_m^{16} = \frac{1}{2(m-2)}\left(1-\frac{1}{2}\pi(m-1)\beta(m)\right).$$

This completes the proof of theorem 1.2.

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